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Weakly Regular Quantum Grammars and Asynchronous Quantum Automata

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Abstract In this paper, we define weakly regular quantum grammars (WRQG), regular quantum grammars (RQG), asynchronous quantum automata (AQA) and synchronous quantum automata (SQA). Moreover, we investigate the relationships between quantum languages generated by weakly quantum regular grammars and by asynchronous quantum automata. At the mean time, we discuss the relationships between regular quantum grammars and synchronous quantum automata.

Keywords Weakly regular quantum grammars · Regular quantum grammars · Asynchronous quantum automata · Synchronous quantum automata

1 Introduction

Quantum computation has become a very intensive research area both in computer science and physics. Benioff [1], in 1982, first raised quantum computing models according to the principles of quantum mechanics which could be at least as powerful as classical ones. Three years later, By re-examining the Church Turing Principle, Deutsch [4], initiated the current computational complexity theory. Additionally, he proposed a model of quantum computer, along with Turing machines in quantum case. Deutsch [5] also introduced quantum networks and investigated some of their properties. Bernstein and Vazirani [2] consolidated the foundation of quantum theories on computation complexity and described an efficient universal quantum computer which could simulate a large class of Quantum Turing Machines. Quantum finite automata (QFA for short), providing a simple theoretical model for quantum

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computers with finite memory (cf. C. Moore, J.P. Crutchfield [14] and A. Kondacs, J. Watrous [10]), aroused considerable studies but attracted few attentions to quantum grammars and the related theories [6, 14, 17]. V. Malyshev [13] defined quantum evolution of words.

C. Moore and J.P. Crutchfield [14] first gave the definition of quantum grammar, and studied the relationships between quantum grammars and quantum automata. It is worth mentioning here, he characterized the unitary properties of quantum grammars by the means of corresponding quantum automata. S. Gudder [6] studied the equivalence between regular quantum grammars and quantum finite automata. In addition, they were furtherly characterized by D.W. Qiu and M.S. Ying [17]. In [12] by R.Q. Lu and H. Zheng, three different types of lattice-valued finite state quantum automata and four distinct operations on them are considered, and their advantages, disadvantages and various properties were discussed as well.

In this study, we mainly show the relationships between WRQG (RQG) and AQA (SQA). These may be useful in quantum communication, quantum automata and quantum information, and also be helpful in proving the equivalence between quantum context-free grammars and quantum pushdown automata. After all, in the classical automata theories, the application cases are abundant [3, 8, 18, 19].

2 Weakly Regular Quantum Grammars

In this section, we give the definition of quantum grammar, slightly different from that of S. Gudder [6] and C. Moore and J.P. Crutchfield [14].

Definition 2.1 An amplitude production ρ over S is an amplitude function

$$\rho : S^* \times S^* \to \mathcal{C}_{[0,1]}$$

such that

(i) $\rho(\varepsilon, \varepsilon) = 1$;
(ii) The set $\overline{\rho} = \{s \in S^* || \rho(s,t)| > 0$ for some $t \in S^*$, where $t \neq s\}$ is finite, where $\rho(s,t)$ is the amplitude that s will be replace by t, and

$$\sum_{t \in S^*} |\rho(s,t)|^2 = 1 \tag{1}$$

for all $s \in S^*$.

Definition 2.2 A weakly regular quantum grammar (WRQG) is a 4-tuple $G = (V, T, P, i)$ such that the following conditions hold

(i) V is a finite set of variables;
(ii) T is a finite set of terminal symbols, and $V \cap T = \emptyset$;
(iii) P is a finite set of amplitude productions over $T \cup V$, for all $\rho \in P$, $\overline{\rho} \subseteq V$ and for all $s \in \overline{\rho}$,

$$|\rho(s,t)| > 0 \tag{2}$$

implies $t = aA$, where $a \in T^*$ and $A \in V \cup \{\varepsilon\}$;

(iv) i is an initial function $i : V \to \mathcal{C}_{[0,1]}$ such that

$$\sum_{A \in V} |i(A)|^2 = 1 \tag{3}$$

where $i(A)$ is the amplitude that A is the initial variable of G.

Definition 2.3 A regular quantum grammar (RQG) is a 4-tuple $G = (V, T, P, i)$ such that the following conditions hold

(i) V, T and i are defined as in Definition 2.1;
(ii) P is a finite set of quantum productions over $T \cup V$, for all $\rho \in P$, $\overline{\rho} = V$ and for all $s \in \overline{\rho}$, $|\rho(s,t)| > 0$ implies $t = aA$, where $a \in T$ and $A \in V \cup \{\varepsilon\}$.

Obviously, from the above two definitions, we know that every regular quantum grammar is weakly regular quantum grammar.

We need to give some definitions and notions to define (weakly) regular languages generated by (weakly) regular quantum grammars. For the similar definitions in the probabilistic case and fuzzy case we refer to [15].

Definition 2.4 Let $r, s \in S^*$ and $k \in N$, then $m(r,s) = k$ if there exist $u_i, v_i \in S^*$, $i = 1, 2, \ldots, k$, where $u_i \neq v_j$ for $i \neq j$, such that

(i) $r = u_i s v_i$ for all $i = 1, 2, \ldots, k$;
(ii) $r = usv$ implies $u = u_i$ and $v = v_i$ for some i, where $i = 1, 2, \ldots, k$, we define $m(r,s) = 0$ if $r \neq usv$ for all $u, v \in S^*$.

We denote $m(r,s) = k$ if and only if r can be expressed in the form usv in exactly k kinds of distinct ways.

Definition 2.5 Let $G = (V, T, P, i)$ be a (weakly) regular quantum grammar. A replacement function of G is a function λ,

$$\lambda : (T \cup V)^* \times \overline{P} \to \mathcal{C}_{[0,1]}$$

such that $\lambda(r) = s$ implies $m(r,s) > 0$. If $\lambda(r) = s$ implies $r = usv$, where $u \in T^*$ and $v \in (T \cup V)^*$, then λ is called leftmost.

We denote $\lambda(r) = s$ if and only if some occurrence of s in r will be replaced. $D(G)$ is the set of all replacement functions of G.

Definition 2.6 Let $r, w, s, t \in S^*$ and $k \in N$, then we define $r \sim^k w \mathrm{mod}(s,t)$ if there exist $u, v \in S^*$ such that $r = usv$, $w = utv$ and $m(us, s) = k$.

We denote $r \sim^k w \mathrm{mod}(s,t)$ if and only if w can be obtained from r by replacing the k-th occurrence of s in r by t. In the sequel, the derivations of quantum grammar are leftmost only.

Definition 2.7 Let $G = (V, T, P, i)$ be a (weakly) regular quantum grammar and $S = V \cup T$.

(i) We define the transition function of productions $f_{(\rho,\lambda)} : (V \cup T)^* \times (V \cup T)^* \to \mathcal{C}_{[0,1]}$ by

$$f_{(\rho,\lambda)}(r, w) = \begin{cases} \rho(\lambda(r), t), & r \sim^k w \mathrm{mod}(\lambda(r), t), \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

(ii) We define the derivation function $f_d : (V \cup T)^* \times (V \cup T)^* \to \mathcal{C}_{[0,1]}$ by

$$f_\varepsilon(r, w) = \begin{cases} 1, & r = w, \\ 0, & r \neq w, \end{cases} \tag{5}$$

and

$$f_{d(\rho,\lambda)}(r, w) = \sum_{x \in (V \cup T)^*} f_d(r, x) f_{(\rho,\lambda)}(x, w) \tag{6}$$

for all $d \in (P \times D(G))^*$, $f_d(r, w)$ is the amplitude that w can be obtained from r by the derivation d.

Definition 2.8 Let $G = (V, T, P, i)$ be a (weakly) regular quantum grammar. The quantum language f_G generated by G over T, $f_G : T^* \to [0, 1]$ is defined by

$$f_G(w) = \left| \sum_{d \in (P \times D(G))^*,\ A \in V} i(A) f_d(A, w) \right|^2 \tag{7}$$

for all $w \in T^*$.

Then, we denote

$$\mathrm{supp}(f_G) = \{w \in T^* | f_G(w) > 0\} \tag{8}$$

as the support set of (weakly) regular quantum language generated by G.

Definition 2.9 Two (weakly) regular quantum language G_1 and G_2 are equivalent if they generated the (weakly) regular quantum language, $f_{G_1}(w) = f_{G_2}(w)$ for all w.

Theorem 2.1 *Let $G_0 = (V_0, T, P_0, i_0)$ be a (weakly) regular quantum grammar. There exists (weakly) regular quantum grammar $G = (V, T, P, A_0)$ such that $f_{G_0}(w) = f_G(w)$ for all w.*

Proof Let $G_0 = (V_0, T, P_0, i_0)$ be a (weakly) regular quantum grammar. We construct a (weakly) regular quantum grammar $G = (V, T, P, i)$, where $V = V_0 \cup \{A_0, A_1\}$, $V_1 = V \setminus \overline{P}_0$, $A_0, A_1 \notin T \cup V_0$. Let a_0 be an arbitrarily fixed element of T, for every $\rho \in P_0$, we define quantum productions ρ' and ρ'' over $T \cup V$, such that for all $s, t \in (T \cup V)^*$

$$\rho'(s,t) = \begin{cases} \rho(s,t), & \text{if } s \in \overline{P}_0, t \in (T \cup V_0)^*, \\ (1 - \sum_{w \in (T \cup V_0)^*} |\rho(s, w)|^2)^{1/2}, & \text{if } s \in \overline{P}_0, t = a_0 A_1, \\ 1, & \text{if } s = t \notin \overline{P}_0 \cup V \text{ or } s \in V_1, t = a_0 A_1, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

and

$$\rho''(s,t) = \begin{cases} \sum_{A \in V_0} i(A) \rho(A, t), & \text{if } s = A_0, t \in (T \cup V_0)^*, \\ (1 - \sum_{A \in V_0} |i(A) \rho(A, t)|^2)^{1/2}, & \text{if } s = A_0, t = a_0 A_1, \\ 1, & \text{if } s = t \neq A_0, \\ 0, & \text{otherwise}. \end{cases} \tag{10}$$

Let $G=(T,V,P,i)$ where P is the collection of all ρ', ρ'' defined above. For $t\in(T\cup V)^*$, amplitude productions ρ in P satisfies $\sum_{s\in(T\cup V)^*}|\rho(s,t)|^2=1$; and

$$i(A)=\begin{cases}1, & A=A_0,\\ 0, & A\neq A_0.\end{cases}\tag{11}$$

By Definitions 2.8 and 2.9, for some $w\in T^*$,

$$\begin{aligned} f_{G_0}(w)&=\left|\sum_{d\in(P_0\times D(G_0))^*,\ A\in V} i_0(A)f_{(G_0)d}(A,w)\right|^2\\ &=\left|\sum_{d\in(P\times D(G))^*} f_{(G)d}(A_0,w)\right|^2=f_G(w).\end{aligned}\tag{12}$$

Moreover, it can be verified that $f_{G_0}(w)=f_G(w)$ for all w. □

Remark 2.1 If a regular quantum grammar $G=(V,T,P,i)$ satisfies the conditions (i). Every $\rho\in P$ satisfies $\sum_{s\in(T\cup V)^*}|\rho(s,t)|^2=1$ for $t\in(T\cup V)^*$; (ii) If $A=A_0$, $i(A)=1$, otherwise $i(A)=0$; and $V\subseteq\overline{P}$; we say that G is a total regular quantum grammar.

3 Asynchronous Quantum Automata

In the classical automata theory [9], a language is regular if and only if the language is recognized by some finite automaton. Up to now, some authors have already studied the relationship between quantum grammars and quantum finite automata [6, 14, 17]. In this section, we discuss the relationships between weakly regular quantum grammars and asynchronous quantum automata.

Definition 3.1 An asynchronous quantum automaton (AQA) is a 6-tuple $\mathcal{M}=(Q,X,Y,\delta,h,g)$ where Q is a finite set of states, X is a finite set of input symbols, and Y is a finite set of output symbols, δ is the amplitude transition function, $\delta: Q\times X\times Y^*\times Q\to\mathcal{C}_{[0,1]}$ where $\delta(q,x,y,q')$ is the amplitude that the next state of $\mathcal{M}$ is q' and output string y is produced when the present state of $\mathcal{M}$ is q and input symbol x is applied, and the amplitude transition satisfies

$$\sum_{q'}|\delta(q,x,y,q')|^2=1,\tag{13}$$

and h is the function of initial states, $h: Q\to\mathcal{C}_{[0,1]}$ such that

$$\sum_{q\in Q}|h(q)|^2=1\tag{14}$$

$h(q)$ is the amplitude that q is the initial state of $\mathcal{M}$. Similarly, g is the final states function $g: Q\to\mathcal{C}_{[0,1]}$ such that

$$\sum_{q\in Q}|g(q)|^2=1\tag{15}$$

$g(q)$ is the amplitude that q is the final state of $\mathcal{M}$.

Each state of $\mathcal{M}$ can be any superposition of states in Q, we denote $|q\rangle$ as the superstition only consisting of q. Let $l_2(Q) = \mathcal{H} = \text{span}\{|q\rangle | q \in Q\}$. Therefore, a global state of $\mathcal{M}$ in the space has a form $|\psi\rangle = \sum_i \alpha_i |q_i\rangle$, and $\sum_i |\alpha_i|^2 = 1$, $\alpha_i \in \mathcal{C}_{[0,1]}$. By Definition 3.1, it's straightforward that $|init\rangle = \sum_i h(q_i)|q_i\rangle$ and $|final\rangle = \sum_i g(q_i)|q_i\rangle$.

Definition 3.2 We define a linear transition operator $U(x, y) : l_2(Q) \to l_2(Q)$ by

$$U(x, y)|\psi\rangle = \sum_i \alpha_i U(x, y)|q_i\rangle \tag{16}$$

and we have

$$U(x, y)|q_i\rangle = \sum_{q_i' \in Q} \delta(q_i, x, y, q_i')|q_i'\rangle \tag{17}$$

where $x \in X$, $y \in Y^*$.

Lemma 3.1 [6] *A linear operator $U(x, y)$ on the $l_2(Q)$ is unitary if and only if it satisfies*

$$\sum_{q'} \delta(q, x, y, q')\delta(q'', x, y, q')^* = \delta_{q,q''}. \tag{18}$$

Definition 3.3 Let $\mathcal{M} = (Q, X, Y, \delta, h, g)$ be an asynchronous quantum automaton. The extended amplitude transition function $\widetilde{\delta}$ of $\mathcal{M}$ is a function $\delta : Q \times X^* \times Y^* \times Q \to \mathcal{C}_{[0,1]}$ defined recursively on the length of x, $x \in X^*$ as follows:

$$\widetilde{\delta}(q', \varepsilon, \varepsilon, q'') = \begin{cases} 1, & \text{if } q' = q'', \\ 0, & \text{if } q' \neq q'', \end{cases} \tag{19}$$

and

$$\widetilde{\delta}(q', ux, y, q'') = \sum_{q \in Q, y = y_1 y_2} \delta(q', u, y_1, q)\widetilde{\delta}(q, x, y_2, q'') \tag{20}$$

where $u \in X$, $x \in X^*$, $y, y_1, y_2 \in Y^*$.

By Definitions 3.2 and 3.3, for an initial state $|q_0\rangle$, and $x = x_1 \cdots x_n$, $x_i \in X$, $y = y_1 \cdots y_n$, $y_i \in Y^*$ we have

$$\begin{aligned} U(x_1 \cdots x_n, y_1 \cdots y_n)|q_0\rangle &= \sum_{q_1 \in Q} \delta(q_i, x, y, q_i')|q_i'\rangle U(x_2 \cdots x_n, y_2 \cdots y_n)|q_1\rangle = \cdots \\ &= \sum_{q_i \in Q} \delta(q_0, x_1, y_1, q_1) \cdots \delta(q_{n-2}, x_{n-1}, y_{n-1}, q_{n-1}) U(x_n, y_n)|q_{n-1}\rangle \\ &= \sum_{q_i \in Q, y = y_1 \cdots y_n} \delta(q_0, x_1, y_1, q_1) \cdots \delta(q_{n-1}, x_n, y_n, q_n)|q_n\rangle \\ &= \sum_{q_i \in Q, y = y_1 \cdots y_n} \widetilde{\delta}(q_0, x, y, q_n)|q_n\rangle. \end{aligned}$$

Then, we define quantum language in the following way:

Definition 3.4 We define the function $f_{\mathcal{M}} : X^* \times Y^* \to [0, 1]$ by

$$f_{\mathcal{M}}(x, y) = |\langle \mathit{final}|U(x, y)|\mathit{init}\rangle|^2$$

i.e.,

$$f_{\mathcal{M}}(x, y) = \left|\sum_{q,q'} h(q)\widetilde{\delta}(q, x, y, q')g(q')\right|^2 \tag{21}$$

where $x \in X^*$, $y \in Y^*$, $f_{\mathcal{M}}(x, y)$ is the probability that $\mathcal{M}$ will produce y when x is applied. We define $f_{\mathcal{M}}$ as the quantum language generated by AQA $\mathcal{M}$.

Let $R(f_{\mathcal{M}})$ be denoted as the set of all output strings of AQA $\mathcal{M}$, i.e., there exists $x \in X^*$ such that $f_{\mathcal{M}}(x, y) \in (0, 1]$, or

$$R(f_{\mathcal{M}}) = \{y \in Y^* | f_{\mathcal{M}}(x, y) \in (0, 1], x \in X^*\} \tag{22}$$

we consider $R(f_{\mathcal{M}})$ as the range of quantum language function $f_{\mathcal{M}}$ for AQA.

Now, we show the relationships between $\mathrm{supp}(f_G)$ and $R(f_{\mathcal{M}})$.

Theorem 3.1 *Let G is a weakly regular quantum grammar. Then there is a asynchronous quantum automata $\mathcal{M}$ such that* $\mathrm{supp}(f_G) = R(f_{\mathcal{M}})$.

Proof Suppose that f_G is a weakly regular quantum language for a weakly regular quantum grammar $G = (V, T, P, i)$, and $\mathrm{supp}(f_G) = \{w \in T^* | f_G(w) > 0\}$. By Theorem 2.1, f_G for some weakly regular quantum grammar $G = (V, T, P, A_0)$ let $A_1, A_2 \notin T \cup V$ and $V_0 = V \cup \{A_1, A_2\}$, we define $\mathcal{M} = (V_0, P, T, \delta, h, g)$ such that for all $\rho \in P$, $r \in T^*$ and $A, A' \in V_0$,

$$\delta(A, \rho, r, A') = \begin{cases} \rho(A, rA'), & \text{if } A, A' \in V, \\ \rho(A, r), & \text{if } A \in V, A' = A_1, \\ 1, & \text{if } A' = A_2 \text{ and } A \in \{A_1, A_2\}, \\ 0, & \text{otherwises}, \end{cases} \tag{23}$$

and we denote A_1 as the unique final state, i.e.,

$$g(A) = \begin{cases} 1, & A = A_1, \\ 0, & A \neq A_0, \end{cases} \tag{24}$$

and A_0 as the unique initial state, i.e.,

$$h(A) = \begin{cases} 1, & A = A_1, \\ 0, & A \neq A_0. \end{cases} \tag{25}$$

By Definitions 2.1 and 3.1, the transition function satisfies

$$\sum_{A'} |\delta(A, \rho, r, A')|^2 = \sum_{A'} |\rho(A, rA')|^2 = 1. \tag{26}$$

Next, we prove $\mathrm{supp}(f_G) \subseteq R(f_{\mathcal{M}})$. By Definitions 2.7 and 2.8, f_G is a weakly regular quantum language, then for each $w \in \mathrm{supp}(f_G)$, $f_G(w) = |\sum_{d\in(P\times D(G))^*} f_d(A_0, w)|^2$, and $f_{d(\rho,\lambda)}(A_0, w) = \sum_{r\in(V\cup T)^*} f_d(A_0, r) f_{(\rho,\lambda)}(r, w)$ such that $f_G(w) > 0$. In other words, there must exist a complete derivation from A_0 to w, for some $r_1, \ldots, r_i \in (V \cup T)^*$, $r_{i-1} \sim^k r_i \operatorname{mod}(\lambda(r_{i-1}), r_i)$ such that $\rho(\lambda(r_{i-1}), r_i) \neq 0$. Therefore, for each $w \in \operatorname{supp}(f_G)$, the construction of AQA $\mathcal{M}$ indicates that

$$\begin{aligned}
f_{\mathcal{M}}(x, w) &= \left|\sum_{A_0, A'} h(A_0)\widetilde{\delta}(A_0, x, w, A')g(A')\right|^2 = |\widetilde{\delta}(A_0, x, w, A')|^2 \\
&= \left|\sum_{A'_1, w=w_1w'} \delta(A_0, x_1, w_1, A'_1)\widetilde{\delta}(A'_1, x', w', A')g(A')\right|^2 = \cdots \\
&= \left|\sum_{A'_1,\ldots,A'_k, w=w_1w_2\cdots w_k} \delta(A_0, x_1, w_1, A'_1)\cdots\delta(A'_k, x_k, w_k, A')\right|^2 \\
&= \left|\sum_{A'_1,\ldots,A'_k, w=w_1w_2\cdots w_k} x_1(A_0, w_1A'_1)\cdots x_k(A'_k, w_kA')\right|^2 \\
&= \left|\sum_{d\in(P\times D(G))^*} f_d(A_0, w)\right|^2 = f_G(w)
\end{aligned}$$

where $D(G) = \{\lambda|\lambda(r_i) = A_i, r_i = w_iA_i\}$ is the set of all replacement functions, which is produced by the derivation from A_0 to w, $x_i \in P$.

Thus, we have $w \in R(f_{\mathcal{M}})$, $\operatorname{supp}(f_G) \subseteq R(f_{\mathcal{M}})$.

Similarly, we prove $R(f_{\mathcal{M}}) \subseteq \operatorname{supp}(f_G)$. For every $w \in R(f_{\mathcal{M}})$, there exists a series of amplitude productions $x_1, \ldots, x_k$ such that $f_{\mathcal{M}}(x_1\cdots x_k, w) > 0$. And from the construction of the AQA, we can conclude $f_G(w) = f_{\mathcal{M}}(x_1\cdots x_k, w)$ following the previous proof. Therefore, $\operatorname{supp}(f_G) = R(f_{\mathcal{M}})$ is valid for some AQA. □

Theorem 3.2 *Let $\mathcal{M}$ be an AQA. Then there is a weakly regular quantum grammar G such that* $\operatorname{supp}(f_G) = R(f_{\mathcal{M}})$.

Proof Suppose that $\mathcal{M} = (Q, X, Y, \delta, h, g)$ is an AQA. We construct a weakly regular quantum grammar $G = (Q, Y, P, h)$, without loss of generality, we assume that $Q \cap Y = \emptyset$, and for all $u \in X$, we define the amplitude production ρ_u over $Y \cup Q$, such that

$$\rho_u(q, r) = \begin{cases} \delta(q, u, y, q'), & \text{if } r = yq', \\ 0, & \text{otherwise,} \end{cases} \tag{27}$$

and for all q satisfies $\sum_r |\rho_u(q, r)|^2 = \sum_{q'} |\delta(q, u, y, q')|^2 = 1$, where $\overline{\rho_u} \in Q$.

Moreover, let ρ_0 be the amplitude productions over $Y \cup Q$, and

$$\rho_0(q, r) = \begin{cases} g(q), & \text{if } r = \varepsilon, \\ 0, & \text{otherwise,} \end{cases} \tag{28}$$

where $\overline{\rho_0} \in Q$. f_G is weakly regular quantum language generated by WRQG G. In the following, we show that $\operatorname{supp}(f_G) = R(f_{\mathcal{M}})$.

To begin with, we prove $\mathrm{supp}(f_G) \subseteq R(f_{\mathcal{M}})$. If each $w \in \mathrm{supp}(f_G)$, then

$$0 < f_G(w) = \left| \sum_{d \in (P \times D(G))^*} h(q) f_d(q, w) \right|^2 = \left| \sum_{r \in (X \cup Y)^*} h(q) f_d(q, r) f_{\rho,\lambda}(r, w) \right|^2. \tag{29}$$

For $w \in \mathrm{supp}(f_G)$, there exist a complete derivation from initial variable to w, i.e., have $r_1, \ldots, r_i \in (V \cup T)^*$, $r_{i-1} \sim^k r_i \mathrm{mod}(\lambda(r_{i-1}), r_i)$ such that $\rho(\lambda(r_{i-1}), r_i) \neq 0$, therefore, (29) can be written as

$$\left| \sum_{r \in (X \cup Y)^*} h(q) f_d(q, r) f_{\rho,\lambda}(r, w) \right|^2 = \left| \sum_{r \in (X \cup Y)^*} h(q) \rho(\lambda(r_1), r_2) \cdots \rho(\lambda(r_k), r_{k+1}) \right|^2. \tag{30}$$

By Definitions 2.5 and 2.6, for a complete derivation, there exist $r_i = \varepsilon$ such that $\rho(\lambda(r_{i-1}), r_i) = g(q)$, otherwise $\rho(\lambda(r_j), r_{j+1}) = \delta(q, x, y, q')$ where $i \neq j$, then

$$\begin{aligned} &\left| \sum_{r \in (X \cup Y)^*} h(q) \rho(\lambda(r_1), r_2) \cdots \rho(\lambda(r_k), r_{k+1}) \right|^2 \\ &\quad = \left| \sum_{q_i} h(q_1) \delta(q_1, x_1, w_1, q_2) \cdots \delta(q_k, x_k, w_k, q_{k+1}) g(q_{k+1}) \right|^2 \\ &\quad = f_{\mathcal{M}}(x_1 \cdots x_k, w_1 \cdots w_k) = f_{\mathcal{M}}(x, w). \end{aligned}$$

Next, we prove $R(f_{\mathcal{M}}) \subseteq \mathrm{supp}(f_G)$. For each $w \in R(f_{\mathcal{M}})$, there exists a $x \in X^*$ such that $f_{\mathcal{M}}(x, w) > 0$, from the previous proof, we can easily get $f_{\mathcal{M}}(x, w) = f_G(w)$, where f_G is weakly regular quantum language generated by the newly constructed grammar. □

4 Regular Quantum Grammar and Synchronous Quantum Automata

In this section, as we discuss before, we will study regular quantum grammar and synchronous quantum automata. C. Moore [14], S. Gudder [6] and D.W. Qiu and M.S. Ying [17] have done some research on quantum finite automata and regular quantum grammars.

Definition 4.1 A synchronous quantum automaton (SQA) is a 5-tuple $\mathcal{M} = (Q, X, Y, \delta, h)$ where Q is a finite set of states, X is a finite set of input symbols, and Y is a finite set of output symbols, δ is an amplitude transition function, $\delta : Q \times X \times Y \times Q \to \mathcal{C}_{[0,1]}$ and the amplitude transition satisfies

$$\sum_{q'} |\delta(q, x, y, q')|^2 = 1 \tag{31}$$

h and g are the same as in Definition 3.1.

Remark 4.1 If for all $q_i \in Q$, $g(q_i) = 1$, then the synchronous quantum automata will reduce to sequential quantum automata [7, 11, 16]. S. Gudder [7] defined quantum sequential machines and gave the conditions for keeping unitarity properties of quantum sequential machines and studied some properties of quantum sequential machines. D.W. Qiu [16] and L.Z. Li [11] showed some properties of sequential quantum machines and solved an open problem proposed by S. Gudder [7].

Definition 4.2 Let $\mathcal{M} = (Q, X, Y, \delta, h)$ be synchronous quantum automata. The extended amplitude transition function $\widetilde{\delta}$ of $\mathcal{M}$ is a function $\delta : Q \times X^* \times Y^* \times Q \to \mathcal{C}_{[0,1]}$ defined recursively on the length of x, and $x \in X^*$, as follows

$$\widetilde{\delta}(q', ux, vy, q'') = \sum_{q \in Q} \delta(q', u, v, q)\widetilde{\delta}(q, x, y, q'') \tag{32}$$

where $u \in X$, $x \in X^*$, $y \in Y^*$ and $|x| = |y|$.

Similar to Definition 3.4, quantum language generated by AQA can be defined in the following way.

Definition 4.3 We define the function $f_{\mathcal{M}} : X^* \times Y^* \to [0, 1]$ by

$$f_{\mathcal{M}}(x, y) = \| \langle \mathit{final} | U(x, y) | \mathit{init} \rangle \|^2 y$$

or,

$$f_{\mathcal{M}}(x, y) = \left| \sum_{q,q'} h(q)\widetilde{\delta}(q, x, y, q')g(q') \right|^2 \tag{33}$$

where $x \in X^*$, $y \in Y^*$, $|x| = |y|$. $f_{\mathcal{M}}(x, y)$ is the probability that $\mathcal{M}$ will produce y when x is applied. We denote $f_{\mathcal{M}}$ as the quantum language generated by SQA $\mathcal{M}$.

Let $R(f_{\mathcal{M}})$ be the set of all output strings of SQA $\mathcal{M}$, i.e., there exist $x \in X^*$, $|x| = |y|$, such that $f_{\mathcal{M}}(x, y) \in (0, 1]$, or

$$R(f_{\mathcal{M}}) = \{y \in Y^* | f_{\mathcal{M}}(x, y) \in (0, 1], x \in X^*, |x| = |y|\} \tag{34}$$

we consider $R(f_{\mathcal{M}})$ as the range of quantum language function $f_{\mathcal{M}}$ for SQA.

Theorem 4.1 *If f_G is regular quantum language, then* $\mathrm{supp}(f_G) = R(f_{\mathcal{M}})$ *for some SQA $\mathcal{M}$.*

Proof The proof is very similar to that of Theorem 3.1. In particular, if $f_G(\varepsilon) > 0$ for $\varepsilon \in Y^*$, then there exists an amplitude production $\rho(A, \varepsilon) \neq 0$, we define $\delta(A, x_0, \varepsilon, A) = \rho(A, \varepsilon)$, where $x_0 \notin X$. We denote $X \cup \{x_0\}$ as the input symbols for SQA. □

Remark 4.2 For a SQA $\mathcal{M}$, $f_{\mathcal{M}}(x, y)$ is the probability that $\mathcal{M}$ will produce y when x is applied, and $|x| = |y|$. If $y = \varepsilon \in Y^*$, is the output string for SQA, then there exists a $x \in X^*$ such that $\widetilde{\delta}(q, x, \varepsilon, q) = \delta(q, x, \varepsilon, q) \neq 0$, although $|x| = |\varepsilon|$. But, it is clear that $\delta(q, x, \varepsilon, q)$ is undefined. Thus, we have $\varepsilon \notin R(f_{\mathcal{M}})$ of SQA.

Theorem 4.2 *Let $R(f_{\mathcal{M}})$ be the set of all output strings of SQA $\mathcal{M}$, then there exists a regular quantum grammar G such that $R(f_{\mathcal{M}}) =$* $\mathrm{supp}(f_G)$ *for $w \in Y^+$.*

Proof Supposed that $R(f_{\mathcal{M}})$ be the set of all outputs string of a SQA $\mathcal{M} = (Q, X, Y, \delta, h)$, which is defined in Definitions 4.1 and 4.2. We construct a regular quantum grammar $G =$

(Q, Y, P, h). Assume that $Q \cap Y \neq \emptyset$, and for all $u \in X$, we define the amplitude productions ρ_u^1 and ρ_u^2 such that

$$\rho_u^1(q,t) = \begin{cases} \delta(q,u,v,q'), & \text{if } t = vq', \\ 0, & \text{otherwise,} \end{cases} \tag{35}$$

and

$$\rho_u^2(q,t) = \begin{cases} \delta(q,u,v,q'), & \text{if } t = v \text{ and } g(q') \neq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{36}$$

where $\overline{\rho_u^1} = \overline{\rho_u^2} = Q$.

For all $q \in Q$, $v \in Y$ and $t \in (Y \cup Q)^*$,

$$\sum_{q \in Q}\{|\rho_u^1(q,t)|^2 + |\rho_u^2(q,t)|^2\} = \sum_{q \in Q} |\delta(q,u,v,q')|^2 = 1. \tag{37}$$

According to the above construction, Remark 4.2 and the proof of Theorem 3.2, there exists a regular quantum grammar G such that $R(f_{\mathcal{M}}) = \mathrm{supp}(f_G)$ for $w \in Y^+$.

Hence, it shows that every quantum languages generated by regular quantum grammars and the quantum languages generated by synchronous quantum automata are equivalent. □

5 Conclusion

The (weakly) regular quantum grammars and (asynchronous) synchronous quantum automata were proposed in this paper. In detail, we proved the equivalence between (weakly) regular quantum grammars and (asynchronous) synchronous quantum automata in the sense of generating languages. In future work, context-free quantum grammars and quantum pushdown automata based on this paper are expected to be discussed.

References

1. Benioff, P.: Quantum mechanical Hamiltonian models of Turing machines. J. Stat. Phys. **29**, 515–546 (1982)
2. Bernstein, E., Vazirani, U.: Quantum complexity theory. SIAM J. Comput. **26**(5), 1411–1473 (1997)
3. Brzozowski, J.A., Negulesc, R.: Automata of asynchronous behaviors. Theor. Comput. Sci. **231**, 113–128 (2000)
4. Deutsch, D.: Quantum theory, the Church Turing principle and the universal quantum computer. Proc. R. Soc. A **400**, 73–90 (1985)
5. Deutsch, D.: Quantum computational networks. Proc. R. Soc. A **425**, 73–90 (1989)
6. Gudder, S.: Basic properties of quantum automata. Found. Phys. **30**, 301–319 (2000)
7. Gudder, S.: Quantum computers. Int. J. Theor. Phys. **39**, 2151–2177 (2000)
8. Halbwachs, N.: About synchronous programming interpretation. Sci. Comput. Programm. **31**, 75–89 (1998)
9. Hopcroft, J., Ullman, J.: An Introduction to Automata Theory Languages and Computation. Addison-Wesley, Reading (1979)
10. Kondacs, A., Watrous, J.: On the power of quantum finite state automata. In: Proceedings of the 38th Annual Symposium on Foundations of Computer Science, pp. 66–75. IEEE Computer Society Press, Los Alamitos (1997)
11. Li, L.Z., Qiu, D.W.: Determination of equivalence between quantum sequential machines. Theor. Comput. Sci. **358**, 65–74 (2006)
12. Lu, R.Q., Zheng, H.: Lattices of quantum automata. Int. J. Theor. Phys. **42**, 1435–1459 (2003)

13. Malyshev, V.: Quantum evolution of words. Theor. Comput. Sci. **273**, 263–269 (2002)
14. Moore, C., Crutchfield, J.: Quantum automata and quantum grammars. Theor. Comput. Sci. **237**, 275–306 (2000)
15. Mordeson, J.N., Malik, D.S.: Fuzzy Automata and Languages: Theory and Applications. Chapman & Hall, London (2002)
16. Qiu, D.W.: Characterizations of sequential quantum machines. Int. J. Theor. Phys. **41**, 811–822 (2002)
17. Qiu, D.W., Ying, M.S.: Characterizations of quantum automata. Found. Phys. **312**, 479–480 (2004)
18. Silver, S.J., Brzozowski, J.A.: True concurrency in models of asynchronous circuit behavior. Form. Methods Syst. Des. **22**, 183–203 (2003)
19. Szmurlo, M., Madelaine, J.: A network of asynchronous micro-servers as a framework for server development. Comput. Netw. ISDN Syst. **29**, 1041–1050 (1997)